

The paper is dedicated to the memory of Professor Nikolai Nekhoroshev.

## GEVREY NORMAL FORM AND EFFECTIVE STABILITY OF LAGRANGIAN TORI

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**ABSTRACT.** A Gevrey symplectic normal form of an analytic and more generally Gevrey smooth Hamiltonian near a Lagrangian invariant torus with a Diophantine vector of rotation is obtained. The normal form implies effective stability of the quasi-periodic motion near the torus.

### 1. INTRODUCTION

The aim of this paper is to obtain a Birkhoff Normal Form (shortly BNF) in Gevrey classes of a Gevrey smooth Hamiltonian near a Kronecker torus  $\Lambda$  with a Diophantine vector of rotation. Such a normal form implies “effective stability” of the quasi-periodic motion near the invariant torus, that is stability in a finite but exponentially long time interval. As in [17, 19, 20] it can be used to obtain a microlocal Quantum Birkhoff Normal Form for the Schrödinger operator  $P_h = -h^2\Delta + V(x)$  near  $\Lambda$  and to describe the semi-classical behavior of the corresponding eigenvalues (resonances).

A Kronecker torus of a smooth Hamiltonian  $H$  in a symplectic manifold of dimension  $2n$  is a smooth embedded Lagrangian submanifold  $\Lambda$ , diffeomorphic to the flat torus  $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$ , which is invariant with respect to the flow  $\Phi^t$  of  $H$ , and such that the restriction of  $\Phi^t$  to  $\Lambda$  is smoothly conjugated to the linear flow  $g_\omega^t(\varphi) := \varphi + t\omega \pmod{2\pi}$  on  $\mathbb{T}^n$  for some  $\omega \in \mathbb{R}^n$ . Hereafter, we suppose that  $\omega$  satisfies the usual Diophantine condition (2.4). Then there is a symplectic mapping  $\chi$  from a neighborhood of the zero section  $\mathbb{T}_0^n := \mathbb{T}^n \times \{0\}$  to a neighborhood of  $\Lambda$  in  $X$  sending  $\mathbb{T}_0^n$  to  $\Lambda$  and such that the Hamiltonian  $H_0 := \chi^*H$  becomes  $H_0(\varphi, I) = H^0(I) + R^0(\varphi, I)$ , where  $\nabla H^0(0) = \omega$ , and the Taylor series of  $R^0$  at  $I = 0$  vanishes (cf. [10], Proposition 9.13). In particular,  $\mathbb{T}_0^n$  is an invariant torus of  $H_0$ , the restriction of the flow of  $H_0$  to  $\mathbb{T}_0^n$  is given by  $g_\omega^t(\varphi) = \varphi + t\omega \pmod{2\pi}$ , and for any  $\alpha, \beta \in \mathbb{N}$ , and any  $N \geq 1$ , we have  $\partial_\varphi^\alpha \partial_I^\beta R^0(\varphi, I) = O_{\alpha, \beta, N}(|I|^N)$ . Our aim is to replace these polynomial estimates by exponential estimates of  $\partial_\varphi^\alpha \partial_I^\beta R^0$

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1991 *Mathematics Subject Classification.* Primary 70H08.

*Key words and phrases.* Birkhoff normal form, Kronecker tori, effective stability, KAM theory.

G.P. partially supported by Agence Nationale de la Recherche, Projet “RESONANCES”: GIP ANR-06-BLAN-0063-03.

of the form  $O_{\alpha,\beta}(\exp(-c/|I|^a))$ ,  $a > 0$ ,  $c > 0$ , in the case when both the Hamiltonian  $H$  and the Kronecker torus are Gevrey smooth. In the case when the Hamiltonian and the torus are analytic a similar BNF has been obtained by Morbidelli and Giorgilli. They have proved as well effective stability of the action near analytic KAM tori and even a super exponential stability of the action [4, 13, 14] for convex Hamiltonians using Nekhoroshev's theory. A simultaneous normal form for a family of Gevrey KAM tori has been obtained in [16, 18].

The existence of large family of Kronecker tori of Diophantine vectors of rotation is given by the classical KAM theorem in the case of real-analytic Hamiltonians satisfying the Kolmogorov non-degeneracy condition and for Gevrey smooth Hamiltonians it has been proved by one of the authors in [18]. Similar results for analytic (Gevrey-smooth) Hamiltonians satisfying the Rüssmann non-degeneracy conditions have been obtained in [22].

## 2. MAIN RESULTS

Let  $X$  be a bounded domain in  $\mathbb{R}^n$ . Fix  $\rho \geq 1$  and a positive constant  $L$ , and denote by  $\mathcal{G}_L^\rho(X)$  the set of all  $C^\infty$ -smooth real-valued functions  $H$  in  $X$  such that

$$\|H\|_L := \sup_{\alpha \in \mathbb{N}^n} \sup_{x \in X} (|\partial_x^\alpha H(x)| L^{-|\alpha|} \alpha!^{-\rho}) < \infty, \quad (2.1)$$

where  $\mathbb{N}$  is the set of non negative integers,  $\alpha! = \alpha_1! \cdots \alpha_n!$  and  $|\alpha| = \alpha_1 + \cdots + \alpha_n$  is the "length" of  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ . A function  $H$  is said to be  $\mathcal{G}^\rho$ -smooth on  $X$  if it satisfies (2.1) with some  $L > 0$ . In the same way, using local coordinates, we define  $\mathcal{G}^\rho$ -smooth functions on a  $\mathcal{G}^\rho$ -smooth manifold  $X$  of dimension  $n$ . Note that the  $\mathcal{G}^1$ -smooth functions in a bounded domain (real-analytic manifold)  $X$  are just the analytic functions in  $X$ . On the other hand, the class of  $\mathcal{G}^\rho$ -smooth function is not quasi-analytic for  $\rho > 1$ ; there exist functions of a compact support which are  $\mathcal{G}^\rho$ -smooth. For more properties of Gevrey smooth functions we refer to [9, 11] and [18, Appendix], where the implicit function theorem and the composition of Gevrey functions is discussed.

When dealing with the KAM theory in Gevrey classes one loses Gevrey regularity in frequencies, and there naturally arise anisotropic Gevrey classes. They are defined as follows. Let  $\rho, \mu \geq 1$  and  $L_1, L_2$  be positive constants. Given a bounded domain  $D \subset \mathbb{R}^n$ , we consider  $\mathbb{A} := \mathbb{T}^n \times D$  provided with the canonical symplectic structure, and denote by  $\mathcal{G}_{L_1, L_2}^{\rho, \mu}(\mathbb{A})$  the set of all  $C^\infty$ -smooth real valued Hamiltonians  $H$  in  $\mathbb{A}$  such that

$$\|H\|_{L_1, L_2} := \sup_{\alpha, \beta \in \mathbb{N}^n} \sup_{(\theta, I) \in \mathbb{A}} \left( |\partial_\theta^\alpha \partial_I^\beta H(\theta, I)| L_1^{-|\alpha|} L_2^{-|\beta|} \alpha!^{-\rho} \beta!^{-\mu} \right) < \infty. \quad (2.2)$$

A Hamiltonian  $H$  in  $\mathbb{A}$  is said to be  $\mathcal{G}^{\rho,\mu}$ -smooth if it belongs to  $\mathcal{G}_{L_1,L_2}^{\rho,\mu}(\mathbb{A})$  for some positive constants  $L_1, L_2$ . The numbers  $\rho \geq 1$  and  $\mu \geq 1$  in (2.2) are called Gevrey exponents and the positive constants  $L_1$  and  $L_2$  are called Gevrey constants. We say that the two pairs of Gevrey constants  $L_1, L_2$  and  $\tilde{L}_1, \tilde{L}_2$  are *equivalent* if there is  $c_1(n, \rho, \mu) > 0$  and  $c_2(n, \rho, \mu) > 0$  such that  $\tilde{L}_1 = c_1(n, \rho, \mu)L_1$  and  $\tilde{L}_2 = c_2(n, \rho, \mu)L_2$ .

Let  $\rho \geq 1$  and let  $X$  be a  $\mathcal{G}^\rho$ -smooth symplectic manifold of dimension  $2n$ . Let  $H$  be a  $\mathcal{G}^\rho$ -smooth Hamiltonian in  $X$ . A  $\mathcal{G}^\rho$ -smooth Kronecker torus of  $H$  of a vector of rotation  $\omega \in \mathbb{R}^n$  is given by a  $\mathcal{G}^\rho$ -smooth embedding  $f : \mathbb{T}^n \rightarrow X$ , such that  $\Lambda = f(\mathbb{T}^n)$  is a Lagrangian submanifold of  $X$  which is invariant with respect to the Hamiltonian flow  $\Phi^t$  of  $H$  and  $\Phi^t \circ f = f \circ g_\omega^t$  for all  $t \in \mathbb{R}$ , i.e. the diagram

$$\begin{array}{ccc} \mathbb{T}^n & \xrightarrow{g_\omega^t} & \mathbb{T}^n \\ \downarrow f & & \downarrow f \\ \Lambda & \xrightarrow{\Phi^t} & \Lambda \end{array} \quad (2.3)$$

is commutative for any  $t \in \mathbb{R}$ . Recall that  $g_\omega^t(\varphi) = \varphi + t\omega \pmod{2\pi}$ . We will suppose that  $\omega$  is  $(\kappa, \tau)$ -Diophantine for some  $\kappa > 0$  and  $\tau > n - 1$ , which means the following:

$$\text{For any } 0 \neq k \in \mathbb{Z}^n, \quad |\langle \omega, k \rangle| \geq \kappa |k|^{-\tau}, \quad (2.4)$$

where  $|k| = \sum_{j=1}^n |k_j|$ . Note that if  $X$  is exact symplectic and  $\Lambda \subset X$  is an embedded submanifold satisfying (2.3) with a Diophantine vector  $\omega$  then  $\Lambda$  is Lagrangian (see [8], Sect. I.3.2). The existence of such tori in  $\mathbb{A} := \mathbb{T}^n \times D$  with vectors of rotation  $\omega$  satisfying (2.4) is provided by the KAM theorem. It follows from [18, Theorems 1.1 and 3.12] and [16], that if  $H \in \mathcal{G}^\rho(\mathbb{A})$  is a “small” (in terms of  $\kappa$ ) real-valued perturbation of a completely integrable Hamiltonian satisfying the Kolmogorov non-degeneracy conditions, then there is a Cantor set  $\Omega_\kappa \in \mathbb{R}^n$  of frequencies satisfying (2.4) and of a positive Lebesgue measure such that for any  $\omega \in \Omega_\kappa$  there is a  $\mathcal{G}^\rho$ -smooth Kronecker torus  $\Lambda_\omega$  with frequency  $\omega$ . In the analytic case ( $\rho = 1$ ) this follows from the classical KAM theorem. Moreover, the family  $\Lambda_\omega$ ,  $\omega \in \Omega_\kappa$ , is  $\mathcal{G}^\mu$ -smooth in Whitney sense, where  $\mu = \rho(\tau + 1) + 1$  when  $\rho > 1$  and  $\mu$  could be any number greater than  $\tau + 2$  when  $\rho = 1$  (see [16, 18, 21]). The main result in this paper is concerned with a Gevrey smooth Birkhoff Normal Form of  $H$  near any Kronecker torus with a Diophantine frequency.

**Theorem 1.** *Let  $\omega \in \mathbb{R}^n$  satisfy the  $(\kappa, \tau)$ -Diophantine condition (2.4) with some  $\kappa > 0$  and  $\tau > n - 1$ . Fix  $\rho \geq 1$  and set  $\mu = \rho(\tau + 1) + 1$ . Let  $H \in \mathcal{G}^\rho(X, \mathbb{R})$  be a real-valued Hamiltonian and let  $\Lambda$  be a  $\mathcal{G}^\rho$ -smooth Kronecker torus of  $H$  of a vector of rotation  $\omega$ . Then there is a neighborhood  $D$  of 0 in  $\mathbb{R}^n$  and a symplectic mapping  $\chi \in \mathcal{G}^{\rho,\mu}(\mathbb{A}, X)$ ,*

where  $\mathbb{A} = \mathbb{T}^n \times D$ , such that  $\chi(\mathbb{T}_0^n) = \Lambda$ , and

$$\begin{cases} H(\chi(\varphi, I)) = H^0(I) + R^0(\varphi, I), \text{ where } H^0 \in \mathcal{G}^\mu(D), R^0 \in \mathcal{G}^{\rho, \mu}(\mathbb{A}), \\ \text{and } \partial_I^\alpha R^0(\varphi, 0) = 0 \text{ for any } \varphi \in \mathbb{T}^n \text{ and } \alpha \in \mathbb{N}^n. \end{cases}$$

In the analytic case ( $\rho = 1$ ), a similar BNF near an elliptic equilibrium point of the Hamiltonian has been obtained by Giorgilli, Delshams, Fontich, Galgani and Simó in [3]. Moreover, effective stability of the action, that is stability of the action in a finite but exponentially long time interval has been proved in [3]. Effective stability near an analytic KAM torus has been investigated by Morbidelli and Giorgilli in [13], [14] and [4]. Combining it with the Nekhoroshev theorem they obtained a super-exponential effective stability of the action near the torus. The Nekhoroshev theory for Gevrey smooth Hamiltonians has been developed by J.-P. Marco and D. Sauzin [12]. As it was mentioned above, if  $H \in \mathcal{G}^\rho(\mathbb{A})$  is a “small” (in terms of  $\kappa$ ) real-valued perturbation of a completely integrable  $\mathcal{G}^\rho$ -smooth Hamiltonian satisfying the Kolmogorov non-degeneracy conditions, then there is a Cantor set  $\Omega_\kappa \subset \mathbb{R}^n$  of frequencies satisfying (2.4) of positive Lebesgue measure such that for any  $\omega \in \Omega_\kappa$  there is a  $\mathcal{G}^\rho$ -smooth Kronecker torus  $\Lambda_\omega$  with frequency  $\omega$ . The family  $\Lambda_\omega$ ,  $\omega \in \Omega_\kappa$ , is  $\mathcal{G}^\mu$ -smooth in Whitney sense, where  $\mu = \rho(\tau + 1) + 1$  if  $\rho > 1$  and  $\mu > \tau + 2$  if  $\rho = 1$  (see [16, 18, 21]). This implies a simultaneous  $\mathcal{G}^{\rho, \mu}$ -smooth BNF of the corresponding Hamiltonian at a family of KAM tori  $\Lambda_\omega$ ,  $\omega \in \tilde{\Omega}_\kappa$ , where  $\tilde{\Omega}_\kappa \subset \Omega_\kappa$  is the set of points of positive Lebesgue density in  $\Omega_\kappa$  [18, Corollary 1.2]. Normal forms for reversible analytic vector fields with an exponentially small error term have been obtained by Iooss and Lombardi [6, 7].

Here we obtain a BNF of any single  $\mathcal{G}^\rho$ -smooth Kronecker torus  $\Lambda$  of the Hamiltonian. This normal form implies effective stability not only of the action but of the *quasi-periodic motion* near  $\Lambda$  as well (cf. [18, Corollary 1.3]). Moreover, our method allows us to keep track on the corresponding Gevrey constants. In the case of KAM tori [16, 18] this yields an uniform bound on the corresponding Gevrey constants with respect to  $\omega \in \Omega_\kappa$ . It could be applied as in [13], [14] and [4] to obtain a super-exponential effective stability of the action near the torus in the case of convex Hamiltonians using the Nekhoroshev theory for Gevrey Hamiltonians developed by J.-P. Marco and D. Sauzin [12]. It seems that this method could be applied to obtain a Gevrey normal form in the case of elliptic tori and near an elliptic equilibrium point of Gevrey smooth Hamiltonians as well as in the case of hyperbolic tori and reversible systems.

The method we use relies on an explicit construction of the generating function of the canonical transformation putting the Hamiltonian in a normal form which allows us to obtain an explicit form of the corresponding homological equation (see Sect. 5). It is different from

those used in [3] and [13] which is based on the formalism of the Lie transform.

It is an interesting question if the exponent  $\mu = \rho(\tau + 1) + 1$  is optimal. As it was mentioned above the same exponent appears in the KAM theorem in Gevrey classes when  $\rho > 1$  and our exponent  $\mu$  is smaller when  $\rho = 1$ , in particular we obtain the same exponent as in the simultaneous BNF of the family of KAM tori  $\Lambda_\omega$ ,  $\omega \in \widetilde{\Omega}_\kappa$  in [18, Corollary 1.2] when  $\rho > 1$ . In the analytic case ( $\rho = 1$ ) there is an heuristic argument of Morbidelli and Giorgilli [14, §3. Discussion] showing that  $\mu = \tau + 2$  should be optimal.

Theorem 1 can be used as in [17, 19, 20] to obtain a microlocal Quantum Birkhoff Normal Form in Gevrey classes for the Schrödinger operator  $P_h = -h^2\Delta + V(x)$  near a Gevrey smooth Kronecker torus  $\Lambda$  of the Hamiltonian  $H(x, \xi) = \|\xi\|^2 + V(x)$ .

### 3. BIRKHOFF NORMAL FORM IN GEVREY CLASSES AND EFFECTIVE STABILITY

We are going to reduce the problem to the case of a Gevrey smooth (real-analytic) Hamiltonian in  $\mathbb{A} = \mathbb{T}^n \times D$  having a Kronecker torus  $\mathbb{T}_0^n = \mathbb{T}^n \times \{0\}$ , where  $D$  is a connected neighborhood of 0 in  $\mathbb{R}^n$  and  $\mathbb{A}$  is provided with the canonical symplectic two-form. By a result of Weinstein there is a symplectic transformation  $\chi_0 : \mathbb{A} \rightarrow X$  such that  $\chi_0(\mathbb{T}_0^n) = \Lambda$  and  $\chi_0 \circ \iota = f$ , where  $\iota(\theta) = (\theta, 0) \in \mathbb{T}_0^n$  for any  $\theta \in \mathbb{T}^n$ . To construct  $\chi_0$  we first find a tubular neighborhood  $U$  of  $\Lambda$  in  $T^*\Lambda$  and a  $\mathcal{G}^\rho$ -smooth symplectic transformation  $F : U \rightarrow X$  which maps the zero section of  $\Lambda$  in  $T^*\Lambda$  to  $\Lambda$ . If  $\rho > 0$  one just follows the proof of Weinstein. In the real-analytic case ( $\rho = 1$ ), we first take a  $C^\infty$ -smooth symplectic map  $F_0$  with this property, which exists by the Weinstein theorem, next we approximate it with a real-analytic one, and then we use a deformation argument of Moser to get  $F$ . Set  $\tilde{f} = F^{-1} \circ f$ . Arguing as in the proof of Proposition 9.13 [10], we obtain a  $\mathcal{G}^\rho$ -smooth symplectic mapping  $\chi_1$  from a bounded neighborhood  $\mathbb{A} = \mathbb{T}^n \times D$  of the torus  $\mathbb{T}_0^n$  in  $T^*\mathbb{T}^n$  to a tubular neighborhood of the zero section of  $\Lambda$  in  $T^*\Lambda$  such that  $\chi_1 \circ \iota = \tilde{f}$ , and we set  $\chi_0 = F \circ \chi_1$ . In particular,  $\chi_0(\mathbb{T}_0^n) = \Lambda$ . Moreover, the pull-back of the Hamiltonian vector field to  $\mathbb{A}$  is globally Hamiltonian and we denote by  $H \in \mathcal{G}^\rho(\mathbb{A}, \mathbb{R})$  its Hamiltonian in  $\mathbb{A}$ . It follows from (2.3) that the restriction of the flow of the Hamiltonian vector field of  $H$  to  $\mathbb{T}_0^n$  is just  $g_\omega^t$ . Moreover,  $H(\theta, 0)$  is constant since the flow is transitive in  $\mathbb{T}_0^n$ , and we take it to be zero. Hence,

$$H(\theta, r) = \langle \omega, r \rangle + \tilde{H}(\theta, r), \quad \tilde{H} \in \mathcal{G}^{\rho, \rho}(\mathbb{A}), \quad \tilde{H}(\theta, r) = O(|r|^2). \quad (3.1)$$

Denote by  $\Gamma(t)$ ,  $t > 0$ , the Gamma function (7.1). Using Remark 7.2, we write the corresponding Gevrey estimates as follows

$$|\partial_\theta^\beta \partial_r^\alpha \tilde{H}(\theta, r)| \leq L_0 L_1^{|\beta|} L_2^{|\alpha|-1} \alpha! \Gamma(\rho|\beta| + 1) \Gamma((\rho - 1)|\alpha| + 1) \quad (3.2)$$

for any  $(\theta, r) \in \mathbb{A}$  and  $\alpha, \beta \in \mathbb{N}^n$ , where  $L_0$ ,  $L_1$  and  $L_2$  are positive constants, and we suppose that  $L_0 \geq 1$ ,  $L_1 \geq 1$  and  $L_2 \geq 1$ .

A smooth function  $g(\theta, I)$  in  $\mathbb{A}' = \mathbb{T}^n \times D'$  is said to be a generating function of a canonical transformation  $\chi : \mathbb{A}' \rightarrow \mathbb{A}$  if

$$\begin{aligned} \text{graph } \chi &:= \{(\chi(\varphi, I); (\varphi, I)) : (\varphi, I) \in \mathbb{A}'\} \\ &= \left\{ \left( \theta, I + \frac{\partial g}{\partial \theta}(\theta, I); \theta + \frac{\partial g}{\partial I}(\theta, I), I \right) \right\}. \end{aligned} \quad (3.3)$$

Without loss of generality we can suppose that  $\kappa \leq 1$  in (2.4). Theorem 1 follows from the following

**Theorem 2.** *Let  $\rho \geq 1$  and  $H \in \mathcal{G}^{\rho, \rho}(\mathbb{A}, \mathbb{R})$ . Suppose that  $H$  satisfies (3.1) and (3.2), where  $\omega \in \mathbb{R}^n$  is  $(\kappa, \tau)$ -Diophantine and  $0 < \kappa \leq 1$  and  $\tau > n - 1$ . Set  $\mu = \rho(\tau + 1) + 1$ . Then there is a neighborhood  $D'$  of 0 in  $\mathbb{R}^n$  and a function  $g \in \mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}', \mathbb{R})$ ,  $g(\theta, I) = O(|I|^2)$  in  $\mathbb{A}' = \mathbb{T}^n \times D'$ , generating a canonical transformation  $\chi \in \mathcal{G}^{\rho, \mu}(\mathbb{A}', \mathbb{A})$ , such that*

$$\begin{cases} H(\chi(\varphi, I)) = H^0(I) + R^0(\varphi, I), \\ \text{where } H^0 \in \mathcal{G}_{C_2}^\mu(D', \mathbb{R}), \quad R^0 \in \mathcal{G}^{\rho, \mu}(\mathbb{A}', \mathbb{R}), \\ \text{and } \partial_I^\alpha R^0(\theta, 0) = 0 \text{ for any } \alpha \in \mathbb{N}^n. \end{cases} \quad (3.4)$$

Moreover, the Gevrey constants  $C_1$  and  $C_2$  are equivalent to  $L_1$  and  $\frac{1}{\kappa} L_0 L_1^{\tau+n+4} L_2$  respectively, i.e.

$$C_1 = c_1(\rho, \tau, n) L_1 \quad \text{and} \quad C_2 = c_2(\rho, \tau, n) \frac{1}{\kappa} L_0 L_1^{\tau+n+4} L_2, \quad (3.5)$$

where  $c_1$  and  $c_2$  are positive constant depending only on  $\rho$ ,  $\tau$  and  $n$ , while  $\kappa$  is the constant in (2.4).

**Remark 3.1.** *We have  $\chi \in \mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}', \mathbb{A})$  and  $R^0 \in \mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}', \mathbb{R})$ , where the Gevrey constants  $C_1$  and  $C_2$  are equivalent to  $L_1^2$  and  $\frac{1}{\kappa} L_0 L_1^{\tau+n+6} L_2$  respectively, i.e.*

$$C_1 = c_1(\rho, \tau, n) L_1^2 \quad \text{and} \quad C_2 = c_2(\rho, \tau, n) \frac{1}{\kappa} L_0 L_1^{\tau+n+6} L_2, \quad (3.6)$$

Theorem 2 and Remark 3.1 will be proved in Sect. 6.

By the Taylor formula of order  $m$  applied to  $R^0(\varphi, I)$  at  $I = 0$  we obtain for any  $\alpha, \beta \in \mathbb{N}^n$ ,  $m \in \mathbb{N}$ , and  $(\varphi, I) \in \mathbb{T}^n \times D'$  the estimate

$$|\partial_\varphi^\alpha \partial_I^\beta R^0(\varphi, I)| \leq A C_1^{|\alpha|} C_2^{|\beta|+m} \alpha!^\rho \beta!^\mu m!^{\mu-1} |I|^m,$$

where  $A > 0$  and the positive constants  $C_1$  and  $C_2$  are as in (3.6). Using Stirling's formula we minimize the right-hand side with respect to  $m \in \mathbb{N}$ . An optimal choice for  $m$  will be

$$m \sim (C_2|I|)^{-\frac{1}{\rho(\tau+1)}},$$

which leads to

$$\begin{aligned} |\partial_\varphi^\alpha \partial_I^\beta R^0(\varphi, I)| &\leq A C_1^{|\alpha|} C_2^{|\beta|} \alpha!^\rho \beta!^{\mu-1} \\ &\times \exp\left(- (C_2|I|)^{-\frac{1}{\rho(\tau+1)}}\right) \end{aligned} \quad (3.7)$$

for any  $\alpha, \beta \in \mathbb{N}^n$  uniformly with respect to  $(\varphi, I) \in \mathbb{T}^n \times D'$ , where  $C_1$  and  $C_2$  are of the form (3.6). This estimate yields effective stability of the quasi-periodic motion near the invariant tori as in [18, Corollary 1.3]).

**3.1. Idea of the Proof of Theorem 2.** Expanding  $\tilde{H}(\theta, r)$  in Taylor series with respect to  $r$  at  $r = 0$  we obtain

$$H(\theta, r) \sim \langle \omega, r \rangle + \sum_{m=2}^{\infty} H_m(\theta, r), \quad H_m(\theta, r) = \sum_{|\alpha|=m} b_\alpha(\theta) r^\alpha. \quad (3.8)$$

It follows from (3.2) that the coefficients  $b_\alpha$  satisfy the following Gevrey type estimates

$$\begin{aligned} |\partial^\beta b_\alpha(\theta)| &= (\alpha!)^{-1} |\partial_\theta^\beta \partial_r^\alpha \tilde{H}(\theta, 0)| \\ &\leq L_0 L_1^{|\beta|} L_2^{|\alpha|-1} \Gamma(\rho|\beta| + 1) \Gamma((\rho - 1)|\alpha| + 1), \end{aligned} \quad (3.9)$$

for any  $\theta \in \mathbb{T}^n$  and any multi-indices  $\alpha, \beta \in \mathbb{N}^n$ ,  $|\alpha| \geq 2$ .

We are looking for a function  $g \in \mathcal{G}^{\rho, \mu}(\mathbb{A}')$ , where  $\mathbb{A}' = \mathbb{T}^n \times D'$  and  $D' \subset \mathbb{R}^n$  is a neighborhood of 0, such that  $g(\theta, 0) = 0$ ,  $\nabla_I g(\theta, 0) = 0$ , and

$$\begin{cases} H(\theta, I + \nabla_\theta g(\theta, I)) = H^0(I) + R(\theta, I), \\ \text{where } H^0 \in \mathcal{G}^\mu(D', \mathbb{R}), \ R \in \mathcal{G}^{\rho, \mu}(\mathbb{A}', \mathbb{R}), \\ \text{and } \partial_I^\alpha R(\theta, 0) = 0 \text{ for any } \alpha \in \mathbb{N}^n. \end{cases} \quad (3.10)$$

If such a function  $g$  exists, and if  $D'$  is sufficiently small, we get by means of the implicit function theorem in anisotropic Gevrey classes [9], [17, Proposition A.2], a function  $\theta(\varphi, I)$  in  $\mathcal{G}^{\rho, \mu}(\mathbb{A}')$  which solves the equation

$$\varphi = \theta + \nabla_I g(\theta, I)$$

with respect to  $\theta \in \mathbb{T}^n$ , and we denote by  $\chi$  the canonical transformation defined by  $g$  by means of (3.3). Hence,

$$(H \circ \chi)(\varphi, I) = H^0(I) + R(\theta(\varphi, I), I).$$

Setting  $R^0(\varphi, I) = R(\theta(\varphi, I), I)$  we obtain  $R^0 \in \mathcal{G}^{\rho, \mu}(\mathbb{A}')$  by the theorem of composition in anisotropic Gevrey classes [17, Proposition A.4], as well as the identities

$$\partial_I^\alpha R^0(\theta, 0) = 0.$$

for any  $\alpha \in \mathbb{N}^n$  and  $\theta \in \mathbb{T}^n$ .

Theorem 2 follows from the following

**Proposition 3.2.** *Let  $\rho \geq 1$ ,  $\tau > n - 1$ , and  $\mu = \rho(\tau + 1) + 1$ . Suppose that the Hamiltonian  $H \in \mathcal{G}^{\rho, \rho}(\mathbb{A}, \mathbb{R})$  satisfies (3.1) and (3.2), where  $\omega$  satisfies (2.4). Then there is a neighborhood  $D'$  of 0 in  $\mathbb{R}^n$  and a function  $g \in \mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}', \mathbb{R})$ ,  $g(\theta, I) = O(|I|^2)$  in  $\mathbb{A}' = \mathbb{T}^n \times D'$ , such that*

$$\begin{cases} H(\theta, I + \nabla_\theta g(\theta, I)) = H^0(I) + R(\theta, I), \\ \text{where } H^0 \in \mathcal{G}_{C_2}^\mu(D', \mathbb{R}), R \in \mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}', \mathbb{R}), \\ \text{and } \partial_I^\alpha R(\theta, 0) = 0 \text{ for any } \alpha \in \mathbb{N}^n, \end{cases} \quad (3.11)$$

where  $C_1$  and  $C_2$  are given by (3.5).

#### 4. WEIGHTED WIENER NORMS

To obtain sharp estimates in Gevrey classes we will use weighted Wiener norms. These norms are well adapted to solve the so called homological equation and they provide a sharp estimate for the product of two functions. Given  $u \in C(\mathbb{T}^n)$ , we denote by  $u_k$ ,  $k \in \mathbb{Z}^n$ , the corresponding Fourier coefficients, and by

$$\langle u \rangle := u_0 = (2\pi)^{-n} \int_{\mathbb{T}^n} u(\varphi) d\varphi$$

the mean value of  $u$  on  $\mathbb{T}^n$ . For any  $s \in \mathbb{R}_+ := [0, +\infty)$  we define the corresponding weighted Wiener norm of  $u$  by

$$S_s(u) := \sum_{k \in \mathbb{Z}^n} (1 + |k|)^s |u_k|,$$

where  $|k| = |k_1| + \dots + |k_n|$ ,  $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ . The weighted Wiener space  $\mathcal{A}^s(\mathbb{T}^n)$ ,  $s \geq 0$ , is defined as the Banach space of all  $u \in C(\mathbb{T}^n)$  such that  $S_s(u) < \infty$  equipped with the norm  $S_s$ . The space  $\mathcal{A}^s(\mathbb{T}^{n-1})$  is a Banach algebra, if  $u, v \in \mathcal{A}^s(\mathbb{T}^n)$  then  $S_s(uv) \leq S_s(u)S_s(v)$ . Moreover, the following relations between Wiener spaces and Hölder spaces hold

$$C^q(\mathbb{T}^n) \hookrightarrow \mathcal{A}^s(\mathbb{T}^n) \hookrightarrow C^s(\mathbb{T}^n),$$

for any  $s \geq 0$  and  $q > s + n/2$ , and the corresponding inclusion maps are continuous. The first relation is a special case of a theorem of Bernstein ( $n = 1$ ) and its generalizations for  $n \geq 2$  [1, Chap. 3, § 3.2]. For more properties of these spaces see [20].



Weighted Wiener spaces are perfectly adapted for solving the homological equation

$$\mathcal{L}_\omega u(\varphi) = f(\varphi) \quad (4.1)$$

where  $\mathcal{L}_\omega := \langle \omega, \frac{\partial}{\partial \theta} \rangle$ . We have the following

**Lemma 4.1.** *Let  $\omega$  satisfy the  $(\kappa, \tau)$ -Diophantine condition (2.4) and let  $s \geq 0$ . Then for any  $f \in \mathcal{A}^{s+\tau}(\mathbb{T}^n)$  such that  $\langle f \rangle = 0$  the homological equation*

$$\mathcal{L}_\omega u = f, \quad \langle u \rangle = 0,$$

*has an unique solution  $u \in \mathcal{A}^s(\mathbb{T}^n)$ , and it satisfies the estimate*

$$S_s(u) \leq \frac{1}{\kappa} S_{s+\tau}(f).$$

*Proof.* Comparing the Fourier coefficients  $u_k$  and  $f_k$ ,  $k \in \mathbb{Z}^n$ , of  $u$  and  $f$  respectively, we get

$$u_k = \frac{f_k}{i \langle k, \omega \rangle}, \quad k \neq 0,$$

and set  $u_0 = 0$ . Then using (2.4) we obtain

$$|u_k| \leq \frac{1}{\kappa} |k|^\tau |f_k| \leq \frac{1}{\kappa} (1 + |k|)^\tau |f_k|, \quad k \neq 0.$$

Since  $f_0 = \langle f \rangle = 0$ , taking the sum with respect to  $k \neq 0$  we get the function  $u$  and the corresponding estimate of  $S_s(u)$ . In this way we obtain an unique solution  $u$  of (4.1) normalized by  $\langle u \rangle = 0$ .  $\square$

In what follows we shall need a sharp estimate of the weighted Wiener norm of the product  $uv$  of two functions  $u, v \in \mathcal{A}^s(\mathbb{T}^n)$ . Let  $[s] \in \mathbb{Z}$  be the integer part of  $s \in \mathbb{R}$  and denote by  $\{s\} = s - [s] \in [0, 1)$  its fractional part.

**Lemma 4.2.** *For any  $s \in \mathbb{R}_+$  and  $u, v \in \mathcal{A}^s(\mathbb{T}^n)$  we have*

$$S_s(uv) \leq 2 \sum_{m=0}^{[s]} \binom{[s]}{m} [S_{s-m}(u)S_m(v) + S_{s-m}(v)S_m(u)].$$

*Proof.* For any  $k \in \mathbb{Z}^n$  we set  $\langle k \rangle := 1 + |k|$ . Obviously,  $\langle k \rangle < \langle l \rangle + \langle k - l \rangle$  for any  $k, l \in \mathbb{Z}^n$ , and we obtain

$$\begin{aligned}
\langle k \rangle^s |(uv)_k| &\leq \sum_{l \in \mathbb{Z}^n} (\langle l \rangle + \langle k - l \rangle)^{[s] + \{s\}} |u_l| |v_{k-l}| \\
&\leq \sum_{l \in \mathbb{Z}^n} \sum_{m=0}^{[s]} \binom{[s]}{m} (\langle l \rangle + \langle k - l \rangle)^{\{s\}} \langle l \rangle^m |u_l| \langle k - l \rangle^{[s]-m} |v_{k-l}| \\
&\leq 2^{\{s\}} \sum_{l \in \mathbb{Z}^n} \sum_{m=0}^{[s]} \binom{[s]}{m} (\langle l \rangle^m |u_l| \langle k - l \rangle^{s-m} |v_{k-l}| \\
&\quad + \langle l \rangle^{m+\{s\}} |u_l| \langle k - l \rangle^{[s]-m} |v_{k-l}|) \\
&= 2^{\{s\}} \sum_{l \in \mathbb{Z}^n} \sum_{m=0}^{[s]} \binom{[s]}{m} (\langle l \rangle^m |u_l| \langle k - l \rangle^{s-m} |v_{k-l}| + \langle l \rangle^{s-m} |u_l| \langle k - l \rangle^m |v_{k-l}|).
\end{aligned}$$

We have used the inequality  $\left| \frac{a+b}{2} \right|^x \leq \max\{a^x, b^x\} \leq a^x + b^x$ , where  $a, b \in \mathbb{N}$  and  $x \geq 0$ . Summing with respect to  $k \in \mathbb{Z}^n$  we prove the claim.  $\square$

A similar inequality can be obtained for the Sobolev  $s$ -norm of  $uv$ , but there appears an additional factor  $2^{s/2}$  coming from the inequality  $(a+b)^2 \leq 2(a^2+b^2)$ , which makes it useless for the estimates in Sect. 6, because it changes the Gevrey constant at any step of the construction.

To get rid of the sum in Lemma 4.2, we consider the modified norms

$$P_s(u) = (s+1)^2 S_s(u), \quad s \geq 0, \quad u \in \mathcal{A}^s(\mathbb{T}^n).$$

If  $f \in \mathcal{A}^{s+\tau}(\mathbb{T}^n)$  and  $\langle f \rangle = 0$ , and if  $u \in \mathcal{A}^s(\mathbb{T}^n)$  is a solution of the homological equation (4.1) such that  $\langle u \rangle = 0$ , then by Lemma 4.1 we obtain

$$P_s(u) = (s+1)^2 S_s(u) \leq \frac{(s+\tau+1)^2}{\kappa} S_{s+\tau}(f) = \frac{1}{\kappa} P_{s+\tau}(f). \quad (4.2)$$

Moreover, for any  $u, v \in \mathcal{A}^s(\mathbb{T}^n)$  we obtain from Lemma 4.2 the following estimate

$$\begin{aligned}
P_s(uv) &\leq 2 \sum_{m=0}^{[s]} \frac{(s+1)^2}{(m+1)^2 (s-m+1)^2} \\
&\quad \times \binom{[s]}{m} [P_{s-m}(u) P_m(v) + P_{s-m}(v) P_m(u)] \\
&\leq \tilde{C} \sup_{0 \leq m \leq [s]} \left\{ \binom{[s]}{m} [P_{s-m}(u) P_m(v) + P_{s-m}(v) P_m(u)] \right\},
\end{aligned} \quad (4.3)$$

where  $\tilde{C} = 16 \sum_{q=1}^{\infty} q^{-2} = 8\pi^2/3$ . Another usefull property of the norm  $P_s(\cdot)$ ,  $s \geq 0$ , is that

$$P_s(\partial^\alpha u) \leq P_{s+|\alpha|}(u)$$

for any  $\alpha \in \mathbb{N}^n$  and  $u \in C^\infty(\mathbb{T}^n)$ .

For any  $p \in \mathbb{N}$  and  $u \in C^\infty(\mathbb{T}^n)$  we set

$$Q_p(u) := \sup_{|\alpha|=p} \sup_{\theta \in \mathbb{T}^n} |\partial_\theta^\alpha u(\theta)|$$

**Lemma 4.3.** *There is a positive constant  $C_0 = C_0(n)$  depending only on the dimension  $n$  such that*

$$Q_{[s]}(u) \leq P_s(u) \leq C_0 (2en)^{[s]} (Q_{[s]+n+2}(u) + Q_0(u)) \quad (4.4)$$

for any  $u \in C^\infty(\mathbb{T}^n)$  and  $s \geq 0$ .

*Proof.* We have

$$P_s(u) \leq C'_0(n)(1+s)^2 \sup_{k \in \mathbb{Z}^n} ((1+|k|)^{[s]+n+2} |u_k|)$$

where  $C'_0(n) := \sum_{k \in \mathbb{Z}^n} (1+|k|)^{-n-1}$ . Integrating by parts we get for any  $p \in \mathbb{N}$  and any  $k \neq 0$  the inequality

$$(1+|k|)^p |u_k| \leq (2n)^p \sup_{1 \leq j \leq n} (|k_j|^p |u_k|) \leq (2n)^p \sup_{|\alpha|=p} \sup_{\theta \in \mathbb{T}^n} |\partial_\theta^\alpha u(\theta)|.$$

Moreover,  $(1+s)^2 \leq 2e^{1+s}$ , and we obtain the second inequality in (4.4) with  $C_0 = 2e^2(2n)^{n+2}C'_0$ . The proof of the first one is straightforward.  $\square$

Consider now the functions  $b_\alpha$  given by (3.8).

**Lemma 4.4.** *We have*

$$P_s(b_\alpha) \leq \tilde{L}_0 L_1^s L_2^{|\alpha|-1} \Gamma(\rho s + (\rho-1)(|\alpha|-2) + 1) \quad (4.5)$$

for any  $s \geq 0$  and any  $\alpha \in \mathbb{N}^n$  with a length  $|\alpha| \geq 2$ , where the Gevrey constants  $L_1 \geq 1$  and  $L_2 \geq 1$  are equivalent to the corresponding Gevrey constants in (3.9) and  $\tilde{L}_0$  is equivalent to  $L_0 L_1^{n+2}$ .

Recall that the positive constant  $\tilde{L}$  is equivalent to  $L$  if there is  $c(n, \rho, \tau) > 0$  such that  $\tilde{L} = c(n, \rho, \tau)L$ .

*Proof.* Using Lemma 4.3 and (3.9), we get

$$P_s(b_\alpha) \leq L_0 L_1^{s+n+2} L_2^{|\alpha|-1} \Gamma(\rho([s] + n + 2) + 1) \Gamma((\rho-1)|\alpha| + 1),$$

where  $L_0$ ,  $L_1$  and  $L_2$  are equivalent to the corresponding constants in (3.9). Note that the function  $\Gamma(t)$  is increasing in the interval  $[3/2, +\infty)$  and that  $x^p \leq e^x p!$  for any  $x \geq 0$  and  $p \in \mathbb{N}$ . Then using (7.2), we obtain

$$\begin{aligned} P_s(b_\alpha) &\leq L_0 L_1^{s+n+2} L_2^{|\alpha|-1} \Gamma(\rho s + (\rho-1)|\alpha| + \rho(n+2) + 2) \\ &\leq e^{\rho(n+4)} p! L_0 L_1^{n+2} (e^\rho L_1)^s (e^{\rho-1} L_2)^{|\alpha|-1} \Gamma(\rho s + (\rho-1)(|\alpha|-2) + 1), \end{aligned}$$

where  $p = ([\rho] + 1)(n + 4)$ . This implies (4.5).  $\square$

## 5. DERIVING THE HOMOLOGICAL EQUATION

We turn now to the construction of the function  $g$ . The idea is to write explicitly the corresponding Taylor series and to prove certain Gevrey estimates for them and then to use a Borel extension theorem in Gevrey classes. Let us expand  $g$  in Taylor series with respect to  $I$  at  $I = 0$ ,

$$g(\theta, I) \sim \sum_{m=2}^{\infty} g_m(\theta, I), \quad g_m(\theta, I) = \sum_{|\alpha|=m} g_{m,\alpha}(\theta) I^\alpha. \quad (5.1)$$

Then we have formally

$$\begin{aligned} & H(\theta, I + \partial g / \partial \theta(\theta, I)) \\ &= \langle \omega, I \rangle + \sum_{m=2}^{\infty} \langle \omega, \frac{\partial g_m}{\partial \theta}(\theta, I) \rangle + \sum_{|\alpha| \geq 2} b_\alpha(\theta) \left( I + \sum_{k=2}^{\infty} \frac{\partial g_k}{\partial \theta}(\theta, I) \right)^\alpha. \end{aligned}$$

We use the the following notations. For any  $a = (a_1, \dots, a_n) \in \mathbb{C}^n$  and  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , we denote by  $a^\alpha$  the product  $a^\alpha := a_1^{\alpha_1} \cdots a_n^{\alpha_n}$ , where by convention  $z^0 = 1$  for any  $z \in \mathbb{C}$ . Let

$$a_k = \{(a_{k,1}, \dots, a_{k,n}) \in \mathbb{C}^n : k \in \mathbb{N}\},$$

be a sequence in  $\mathbb{C}^n$ . Fix  $\alpha \in \mathbb{N}^n$  of length  $|\alpha| \geq 2$ , and recall the following power series expansion in  $\mathbb{C}[[X]]$  (c.f. (4.7) in [5])

$$\begin{aligned} \left( \sum_{k=1}^{\infty} a_k X^k \right)^\alpha &:= \left( \sum_{k=1}^{\infty} a_{k,1} X^k \right)^{\alpha_1} \cdots \left( \sum_{k=1}^{\infty} a_{k,n} X^k \right)^{\alpha_n} \\ &= \sum_{m=|\alpha|}^{\infty} A_{\alpha,m} X^m, \end{aligned}$$

where

$$A_{\alpha,m} = \sum \frac{\alpha!}{\alpha^1! \cdots \alpha^{m-1}!} a_1^{\alpha^1} \cdots a_{m-1}^{\alpha^{m-1}},$$

and the sum is over the set  $\mathbb{N}(\alpha, m)$  of all multi-indices

$$(\alpha^1, \dots, \alpha^{m-1}) \in \underbrace{\mathbb{N}^n \times \cdots \times \mathbb{N}^n}_{m-1}$$

such that

$$\begin{cases} \alpha^1 + \cdots + \alpha^{m-1} = \alpha, \\ 1 \cdot |\alpha^1| + 2 \cdot |\alpha^2| + \cdots + (m-1) \cdot |\alpha^{m-1}| = m. \end{cases} \quad (5.2)$$

Notice that if  $\alpha^1 + \cdots + \alpha^j = \alpha$ ,  $1 \cdot |\alpha^1| + \cdots + j \cdot |\alpha^j| = m$ , and  $\alpha^j \neq 0$ , then  $j \leq m-1$  since  $|\alpha| \geq 2$ . Hence, for any  $\alpha \in \mathbb{N}^n$  with length  $|\alpha| \geq 2$  we obtain

$$\left( I + \sum_{k=2}^{\infty} \frac{\partial g_k}{\partial \theta}(\theta, I) \right)^\alpha = \sum_{m=|\alpha|}^{\infty} A_{\alpha,m}(\theta, I),$$

where  $A_{\alpha,m}(\theta, I)$  is a homogeneous polynomial with respect to  $I$  of degree  $m$  of the form

$$A_{\alpha,m}(\theta, m) = \sum \frac{\alpha!}{\alpha^1! \alpha^2! \dots \alpha^{m-1}!} I^{\alpha^1} \\ \times \left( \frac{\partial g_2}{\partial \theta}(\theta, I) \right)^{\alpha^2} \dots \left( \frac{\partial g_{m-1}}{\partial \theta}(\theta, I) \right)^{\alpha^{m-1}},$$

and the sum is taken over the set of multi-indices  $\mathbb{N}(\alpha, m)$ . Summing with respect to  $\alpha$  we get formally

$$H(\theta, I + \partial g / \partial \theta(\theta, I)) \\ = \langle \omega, I \rangle + \sum_{m=2}^{\infty} \left( \langle \omega, \frac{\partial g_m}{\partial \theta}(\theta, I) \rangle + B_m(\theta, I) \right), \quad (5.3)$$

where  $B_m(\cdot, I)$  is a homogeneous polynomial of degree  $m \geq 2$  with respect to  $I$  of the form

$$B_m(\theta, I) = \sum \frac{\alpha!}{\alpha^1! \alpha^2! \dots \alpha^{m-1}!} b_{\alpha}(\theta) I^{\alpha^1} \\ \times \left( \frac{\partial g_2}{\partial \theta}(\theta, I) \right)^{\alpha^2} \dots \left( \frac{\partial g_{m-1}}{\partial \theta}(\theta, I) \right)^{\alpha^{m-1}}. \quad (5.4)$$

The index set of the sum above is

$$\mathbb{N}(m) := \bigcup_{2 \leq |\alpha| \leq m} \mathbb{N}(\alpha, m)$$

and it consists of all the multi-indices

$$(\alpha^1, \alpha^2, \dots, \alpha^{m-1}) \in \underbrace{\mathbb{N}^n \times \dots \times \mathbb{N}^n}_{m-1}$$

such that

$$1 \cdot |\alpha^1| + 2 \cdot |\alpha^2| + \dots + (m-1) \cdot |\alpha^{m-1}| = m.$$

Note that  $\alpha = \alpha^1 + \alpha^2 + \dots + \alpha^{m-1}$  in (5.4).

For any  $m \geq 2$  we obtain from (3.10) the following homological equation

$$\mathcal{L}_{\omega} g_m(\theta, I) + B_m(\theta, I) = R_m(I), \quad \langle g_m(\cdot, I) \rangle = 0, \quad (5.5)$$

where

$$R_m(I) = \langle B_m(\cdot, I) \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} B_m(\theta, I) d\theta.$$

Denote by  $(B_m)_k(I)$ ,  $k \in \mathbb{Z}^n$ , the Fourier coefficients of  $B_m(\theta, I)$ . Then  $(B_m)_0(I) = R_m(I)$  is a homogeneous polynomial of degree  $m$  with

respect to  $I$ . Since  $\omega$  satisfies (2.4)

$$g_m(\theta, I) = - \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{(B_m)_k(I)}{i \langle \omega, k \rangle} e^{i \langle k, \theta \rangle}, \quad (5.6)$$

and it is a homogeneous polynomial of degree  $m$  with respect to  $I$  with smooth coefficients  $g_{m,\alpha}(\theta)$ ,  $|\alpha| = m$ . Our aim is to obtain Gevrey estimates for  $g_{m,\alpha}(\theta)$ .

## 6. GEVREY ESTIMATES

We are going to show that there are positive constants  $C_1$  and  $C_2$  depending on the constants  $\tilde{L}_0$ ,  $L_1$  and  $L_2$  in (4.5) such that for any  $\alpha \in \mathbb{N}^n$  with length  $m = |\alpha| \geq 2$  and for any  $\beta \in \mathbb{N}^n$  we have

$$\sup_{\theta \in \mathbb{T}^n} |\partial^\beta g_{m,\alpha}(\theta)| \leq C_1^{|\beta|} C_2^{|\alpha|-1} \beta!^\rho \alpha!^{\mu-1}, \quad (6.1)$$

where  $\mu = \rho(\tau + 1) + 1$  (see the statement of Theorem 2). Consider for any  $m \geq 2$  the solution  $(g_m, R_m)$  of the homological equation (5.5), where  $\omega$  is  $(\kappa, \tau)$ -Diophantine,  $0 < \kappa \leq 1$  and  $\tau > n - 1$ . Denote the unit poly-disc in  $\mathbb{C}^n$  by  $\mathbb{D}^n$ , i.e.  $I = (I_1, \dots, I_n) \in \mathbb{C}^n$  belongs to  $\mathbb{D}^n$  if  $|I_j| \leq 1$  for any  $1 \leq j \leq n$ .

**Proposition 6.1.** *There is  $A_0 = A_0(n, \rho, \tau) \geq 1$  depending only on  $n$ ,  $\rho$  and  $\tau$ , such that*

$$\text{for } C_1 = e^\rho L_1 \text{ and for any } C_2 \geq \frac{1}{\kappa} A_0 \tilde{L}_0 L_1^{\tau+2} L_2 \quad (6.2)$$

*the following estimates hold*

$$\sup_{I \in \mathbb{D}^n} P_s(B_m(\cdot, I)) \leq B_0 \tilde{L}_0 L_1^2 L_2 C_1^s C_2^{m-2} \Gamma(\rho s + (\mu - 1)(m - 2)) \quad (6.3)$$

*for  $m \geq 3$  and any  $s \in \mathbb{R}_+$ , and*

$$\sup_{I \in \mathbb{D}^n} P_s(g_m(\cdot, I)) \leq C_1^s C_2^{m-1} \Gamma(\rho s + (\mu - 1)(m - 1) - \rho), \quad (6.4)$$

*for  $m \geq 2$  and any  $s \in \mathbb{R}_+$ , where  $\tilde{L}_0$ ,  $L_1$  and  $L_2$  are the corresponding Gevrey constants in (4.5) and  $B_0 = B_0(n, \rho, \tau) \geq 1$ .*

Note that  $(\mu - 1)(m - 1) - \rho \geq \rho(\tau + 1) - \rho > 1$ . We are going to prove Proposition 6.1 by recurrence with respect to  $m \geq 2$ . For  $m = 2$  we obtain from (5.5) the equation

$$\mathcal{L}_\omega g_2(\theta, I) = R_2(I) - B_2(\theta, I).$$

Moreover, (5.2) and (5.4) imply

$$B_2(\theta, I) = \sum_{|\alpha|=2} b_\alpha(\theta) I^\alpha.$$

For any  $I \in \mathbb{D}^n$  we have by (4.5)

$$P_s(B_2(\cdot, I)) \leq \sum_{|\alpha|=2} P_s(b_\alpha) \leq n^2 \tilde{L}_0 L_1^s L_2 \Gamma(\rho s + 1).$$

Then using (4.2) we obtain

$$P_s(g_2(\cdot, I)) \leq \frac{1}{\kappa} P_{s+\tau}(B_2(\cdot, I)) \leq n^2 L_1^s \left( \frac{1}{\kappa} \tilde{L}_0 L_1^\tau L_2 \right) \Gamma(\rho s + \rho\tau + 1).$$

On the other hand,

$$\Gamma(\rho s + \rho\tau + 1) = (\rho s + \rho\tau) \Gamma(\rho s + \rho\tau) \leq e^{\rho s + \rho\tau} \Gamma(\rho s + \rho\tau),$$

and we obtain

$$P_s(g_2(\cdot, I)) \leq C_1^s C_2 \Gamma(\rho s + (\mu - 1) - \rho)$$

for any  $I \in \mathbb{D}^n$ , where  $C_1 = e^\rho L_1$ ,  $C_2 \geq \frac{1}{\kappa} A_0 \tilde{L}_0 L_1^\tau L_2$  and  $A_0 \geq n^2 e^{\rho\tau}$ .

Fix  $m \geq 3$  and suppose that the estimates (6.4) hold for any  $p < m$  and any  $s \geq 0$ . We are going first to estimate  $P_s(B_m(\cdot, I))$ ,  $I \in \mathbb{D}^n$ , for any  $s \geq 0$ . Using the inductive assumption we get

**Lemma 6.2.** *Let  $p \geq 1$  and  $2 \leq m_k \leq m - 1$ , where  $k = 1, \dots, p$ . Set*

$$M_p = m_1 + \dots + m_p - p.$$

*Then for any  $\delta \in (0, \mu - 1)$  there is a constant  $C_0(\delta, \mu) \geq 1$  such that*

$$\begin{aligned} \sup_{I \in \mathbb{D}^n} P_s \left( \frac{\partial g_{m_1}}{\partial \theta_{j_1}}(\cdot, I) \dots \frac{\partial g_{m_p}}{\partial \theta_{j_p}}(\cdot, I) \right) &\leq C_0(\delta, \mu)^{p-1} C_1^{p+s} C_2^{M_p} \\ &\times \left( \frac{M_p!}{(m_1 - 1)! \dots (m_p - 1)!} \right)^{1+\delta-\mu} \Gamma(\rho s + (\mu - 1)M_p). \end{aligned} \quad (6.5)$$

**Remark 6.3.** *If  $p \geq 2$  then*

$$\frac{M_p!}{(m_1 - 1)! \dots (m_p - 1)!} \geq M_p.$$

**Remark 6.4.** *Recall that  $\mu = \rho(\tau + 1) + 1 > 3$ . We shall fix later  $\delta = \mu - 2 > 1$ .*

*Proof of Lemma 6.2.* We are going to prove (6.5) by induction with respect to  $p \geq 1$ . For  $p = 1$ , we have

$$P_s \left( \frac{\partial g_{m_1}}{\partial \theta_j} \right) \leq P_{s+1}(g_{m_1}) \leq C_1^{s+1} C_2^{m_1-1} \Gamma(\rho s + (\mu - 1)(m_1 - 1)) \quad (6.6)$$

in view of (6.4). Set

$$F_p(\theta, I) = \frac{\partial g_{m_1}}{\partial \theta_{j_1}}(\theta, I) \dots \frac{\partial g_{m_p}}{\partial \theta_{j_p}}(\theta, I).$$

Now take  $p = 2$  and  $2 \leq m_1, m_2 \leq m - 1$ , and fix  $I \in \mathbb{D}^n$ . Using (4.3) and (6.6) we obtain

$$\begin{aligned} P_s(F_2(\cdot, I)) &\leq \tilde{C} C_1^{s+2} C_2^{m_1+m_2-2} \max_{0 \leq q \leq [s]} \left[ \binom{[s]}{q} \right. \\ &\times \left( \Gamma(\rho(s-q) + (\mu-1)(m_1-1)) \Gamma(\rho q + (\mu-1)(m_2-1)) \right. \\ &\left. \left. + \Gamma(\rho(s-q) + (\mu-1)(m_2-1)) \Gamma(\rho q + (\mu-1)(m_1-1)) \right) \right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\Gamma(\rho(s-q) + (\mu-1)(m_1-1)) \Gamma(\rho q + (\mu-1)(m_2-1)) \\ &= \Gamma(\rho s + (\mu-1)(m_1+m_2-2)) \\ &\times B(\rho(s-q) + (\mu-1)(m_1-1), \rho q + (\mu-1)(m_2-1)), \end{aligned}$$

where  $B(x, y)$ ,  $x, y > 0$ , is the Beta function (7.3). Recall that  $B(x, y)$  is decreasing with respect to both variables  $x > 0$  and  $y > 0$ . Then using (7.4) we get for any  $\delta \in (0, \mu-1)$  the inequalities

$$\begin{aligned} &B(\rho(s-q) + (\mu-1)(m_1-1), \rho q + (\mu-1)(m_2-1)) \\ &\leq B(2\rho(s-q) + \delta, 2\rho q + \delta)^{1/2} \\ &\times B(2(\mu-1)(m_1-1) - \delta, 2(\mu-1)(m_2-1) - \delta)^{1/2} \\ &\leq B(2(s-q) + \delta, 2q + \delta)^{1/2} \\ &\times B(2(\mu-1-\delta)(m_1-1) + \delta, 2(\mu-1-\delta)(m_2-1) + \delta)^{1/2}. \end{aligned}$$

Moreover, Lemma 7.1 implies

$$\begin{aligned} &B(2(s-q) + \delta, 2q + \delta) \\ &\leq B(2([s]-q) + \delta, 2q + \delta) \leq C'(\delta) \binom{[s]}{q}^{-2}. \end{aligned} \tag{6.7}$$

as well as

$$\begin{aligned} &B(2(\mu-1-\delta)(m_1-1) + \delta, 2(\mu-1-\delta)(m_2-1) + \delta) \\ &\leq C'(\delta, \mu) \binom{m_1+m_2-2}{m_1-1}^{2(1+\delta-\mu)}. \end{aligned} \tag{6.8}$$



Hence, for any non-negative integer  $0 \leq q \leq [s]$  we have

$$\begin{aligned} & \binom{[s]}{q} B(\rho(s-q) + (\mu-1)(m_1-1), \rho q + (\mu-1)(m_2-1)) \\ & \leq C''(\delta, \mu) \binom{m_1+m_2-2}{m_1-1}^{1+\delta-\mu}. \end{aligned}$$

In the same way we estimate the quantity

$$\Gamma(\rho(s-q) + (\mu-1)(m_2-1))\Gamma(\rho q + (\mu-1)(m_1-1)).$$

Finally we obtain

$$\begin{aligned} P_s(F_2(\cdot, I)) & \leq C_0(\delta, \mu) C_1^{s+2} C_2^{m_1+m_2-2} \\ & \times \Gamma(\rho s + (\mu-1)(m_1+m_2-2)) \left( \frac{(m_1+m_2-2)!}{(m_1-1)!(m_2-1)!} \right)^{1+\delta-\mu}, \end{aligned}$$

where

$$C_0(\delta, \mu) = \max\{2\tilde{C}C''(\delta, \mu), 1\} \geq 1. \quad (6.9)$$

The proof follows by recurrence with respect to  $p$  setting  $F_p = F_{p-1} \frac{\partial g_{m_p}}{\partial \theta_{j_p}}$  and then using (6.5) for  $F_{p-1}$ , (6.6), and the above argument. At any step the constant  $C_0(\delta, \mu)$  is given by (6.9).  $\square$

In the same way as above, using (4.5), we get

**Lemma 6.5.** *Let  $p \geq 1$  and  $2 \leq m_k \leq m-1$ , where  $k = 1, \dots, p$ , and let  $\alpha \in \mathbb{N}^n$  with  $|\alpha| \geq 2$ . Set  $M_p = m_1 + \dots + m_p - p$  and  $C_1 = e^\rho L_1$ . Then for any  $0 < \delta < \mu-1$  and  $I \in \mathbb{D}^n$  we have*

$$\begin{aligned} & P_s \left( b_\alpha \frac{\partial g_{m_1}}{\partial \theta_{j_1}}(\cdot, I) \dots \frac{\partial g_{m_p}}{\partial \theta_{j_p}}(\cdot, I) \right) \\ & \leq K_0 \tilde{L}_0 (e^{\mu-1} L_2)^{|\alpha|-1} C_0^{p-1} C_1^{p+s} C_2^{M_p} \Gamma(\rho s + (\mu-1)(M_p + |\alpha| - 2)) \\ & \times \left( \frac{M_p!}{(m_1-1)! \dots (m_p-1)!} \right)^{1+\delta-\mu} \binom{M_p + |\alpha| - 2}{|\alpha| - 2}^{1+\delta-\mu}, \end{aligned}$$

where  $C_0 = C_0(\delta, \mu) \geq 1$  is given by (6.9) and  $K_0 = K_0(n, \delta, \mu) \geq 1$ .

*Proof.* To simplify the notations we will write below  $M$  instead of  $M_p$ .

Set as above  $F_p(\theta, I) = \frac{\partial g_{m_1}}{\partial \theta_{j_1}}(\theta, I) \dots \frac{\partial g_{m_p}}{\partial \theta_{j_p}}(\theta, I)$  and fix  $I \in \mathbb{D}^n$ . First

suppose that  $|\alpha| = 2$ . Using (4.3) and Lemma 6.2, we obtain

$$\begin{aligned} & P_s(b_\alpha F_p(\cdot, I)) \\ & \leq \tilde{C} C_0^{p-1} C_1^p C_2^M \left( \frac{M_p!}{(m_1-1)! \cdots (m_p-1)!} \right)^{1+\delta-\mu} \\ & \quad \times \max_{0 \leq q \leq [s]} \left[ \binom{[s]}{q} \left( P_q(b_\alpha) \Gamma(\rho(s-q) + (\mu-1)M) C_1^{s-q} \right. \right. \\ & \quad \left. \left. + P_{s-q}(b_\alpha) \Gamma(\rho q + (\mu-1)M) C_1^q \right) \right]. \end{aligned}$$

For  $q = 0$  we have

$$P_0(b_\alpha) \Gamma(\rho s + (\mu-1)M) \leq \tilde{L}_0 L_2 \Gamma(\rho s + (\mu-1)M).$$

For  $q \geq 1$  the estimates (4.5) imply

$$P_q(b_\alpha) \leq \tilde{L}_0 L_1^q L_2 \Gamma(\rho q + 1) \leq \tilde{L}_0 (e^\rho L_1)^q L_2 \Gamma(\rho q) = \tilde{L}_0 C_1^q L_2 \Gamma(\rho q),$$

and we obtain

$$\begin{aligned} & P_q(b_\alpha) \Gamma(\rho(s-q) + (\mu-1)M) \\ & \leq \tilde{L}_0 C_1^q L_2 \Gamma(\rho s + (\mu-1)M) B(\rho q, \rho(s-q) + (\mu-1)M). \end{aligned}$$

On the other hand,  $\rho \geq 1$ ,  $(\mu-1)M \geq \mu-1 = \rho(\tau+1) > 2$ , and  $B(x, y)$  is decreasing with respect to both variables  $x > 0$  and  $y > 0$ , hence, we get

$$\begin{aligned} & B(\rho q, \rho(s-q) + (\mu-1)M) \\ & \leq B(q, [s] - q + 1) = \frac{(q-1)!([s]-q)!}{[s]!} < \binom{[s]}{q}^{-1}. \end{aligned}$$

In the same way we estimate the second term of the sum above. For  $0 \leq q < [s]$  we use the same argument as above. For  $q = [s]$  we get

$$P_{\{s\}}(b_\alpha) \leq \tilde{L}_0 L_1^{\{s\}} L_2 \Gamma(\rho\{s\} + 1) \leq \tilde{L}_0 L_1^{\{s\}} L_2 \Gamma(\rho + 1),$$

since  $\rho + 1 \geq 2$  (see the argument below), and we obtain

$$\begin{aligned} & P_{\{s\}}(b_\alpha) \Gamma(\rho[s] + (\mu-1)M) \\ & \leq \Gamma(\rho + 1) \tilde{L}_0 L_1^{\{s\}} L_2 \Gamma(\rho s + (\mu-1)M). \end{aligned}$$

This proves the claim for  $|\alpha| = 2$ .

Let  $|\alpha| \geq 3$ . Recall that  $\Gamma(t)$  is increasing in the interval  $[2, +\infty)$ ,  $\Gamma(t) \leq 1$  for  $t \in [1, 2]$ , and  $\Gamma(1) = \Gamma(2) = 1$  (see Sect. 7). Hence,  $\Gamma(t_1) \leq \Gamma(t_2)$  if  $1 \leq t_1 \leq t_2$  and  $t_2 \geq 2$ . Since  $\mu-1 = \rho(\tau+1) > 2$  and  $|\alpha|-2 \geq 1$ , this allows us to replace  $\rho-1$  by  $\mu-1$  in (4.5), and we obtain

$$\begin{aligned} & P_s(b_\alpha) \leq \tilde{L}_0 L_1^s L_2^{|\alpha|-1} \Gamma(\rho s + (\mu-1)(|\alpha|-2) + 1) \\ & \leq \tilde{L}_0 (e^\rho L_1)^s (e^{\mu-1} L_2)^{|\alpha|-1} \Gamma(\rho s + (\mu-1)(|\alpha|-2)) \end{aligned} \tag{6.10}$$

for any  $\rho \geq 1$  and  $s \geq 0$ . Using (4.3) and Lemma 6.2, we obtain

$$\begin{aligned}
& P_s(b_\alpha F_p(\cdot, I)) \\
& \leq \tilde{C} \tilde{L}_0 C_0^{p-1} C_1^{s+p} (e^{\mu-1} L_2)^{|\alpha|-1} C_2^M \left( \frac{M!}{(m_1-1)! \cdots (m_p-1)!} \right)^{1+\delta-\mu} \\
& \times \max_{0 \leq q \leq [s]} \left[ \binom{[s]}{q} (\Gamma(\rho q + (\mu-1)(|\alpha|-2)) \Gamma(\rho(s-q) + (\mu-1)M) \right. \\
& \left. + \Gamma(\rho(s-q) + (\mu-1)(|\alpha|-2)) \Gamma(\rho q + (\mu-1)M)) \right].
\end{aligned}$$

Recall that  $|\alpha| \geq 3$ , hence  $\rho q + (\mu-1)(|\alpha|-2) \geq \mu-1 = \rho(\tau+1) > 2$ . We have

$$\begin{aligned}
& \Gamma(\rho q + (\mu-1)(|\alpha|-2)) \Gamma(\rho(s-q) + (\mu-1)M) \\
& = \Gamma(\rho s + (\mu-1)(M + |\alpha|-2)) \\
& \times B(\rho q + (\mu-1)(|\alpha|-2), \rho(s-q) + (\mu-1)M).
\end{aligned}$$

Using (7.4) we get

$$\begin{aligned}
& B(\rho q + (\mu-1)(|\alpha|-2), \rho(s-q) + (\mu-1)M) \\
& \leq B(2\rho q + \delta, 2\rho(s-q) + \delta)^{1/2} \\
& \times B(2(\mu-1)(|\alpha|-2) - \delta, 2(\mu-1)M - \delta)^{1/2}.
\end{aligned}$$

Moreover, using Lemma 7.1 we get as above

$$\begin{aligned}
& B(2\rho q + \delta, 2\rho(s-q) + \delta) \leq B(2q + \delta, 2([s]-q) + \delta) \\
& \leq C'(\delta) \binom{[s]}{q}^{-2},
\end{aligned}$$

and

$$\begin{aligned}
& B(2(\mu-1)(|\alpha|-2) - \delta, 2(\mu-1)M - \delta) \\
& \leq B(2(\mu-1-\delta)(|\alpha|-2) + \delta, 2(\mu-1-\delta)M + \delta) \\
& \leq C'(\delta, \mu) \left( \frac{M + |\alpha| - 2}{|\alpha| - 2} \right)^{2(1+\delta-\mu)}.
\end{aligned}$$

In the same way we estimate the second term. This proves the Lemma taking  $K_0 = K_0(n, \delta, \mu) \geq 1$  sufficiently large.  $\square$

From now on we fix  $\delta = \mu - 2 = \rho(\tau + 1) - 1 \geq \tau > 1$ . We return to the proof of (6.3) and (6.4). First we shall estimate  $P_s(B_m(\cdot, I))$  for

$I \in \mathbb{D}^n$  and  $m \geq 3$ . In view of (5.4) we obtain

$$P_s(B_m(\cdot, I)) \leq \sum_{2 \leq |\alpha| \leq m} Q_\alpha(I),$$

where

$$\begin{aligned} Q_\alpha(I) &= \sum_{(\alpha^1, \dots, \alpha^{m-1}) \in \mathbb{N}(\alpha, m)} \frac{\alpha!}{\alpha^1! \dots \alpha^{m-1}!} \\ &\times P_s \left( b_\alpha \left( \frac{\partial g_2}{\partial \theta}(\cdot, I) \right)^{\alpha^2} \dots \left( \frac{\partial g_{m-1}}{\partial \theta}(\cdot, I) \right)^{\alpha^{m-1}} \right). \end{aligned} \quad (6.11)$$

Consider more closely the index set  $\mathbb{N}(\alpha, m)$ , where  $|\alpha| \geq 2$ . Recall from (5.2) that  $(\alpha^1, \dots, \alpha^{m-1}) \in (\mathbb{N}^n)^{m-1}$  belongs to  $\mathbb{N}(\alpha, m)$  if and only if

$$\begin{cases} \alpha^1 + \dots + \alpha^{m-1} = \alpha, \text{ and} \\ 1 \cdot |\alpha^1| + 2 \cdot |\alpha^2| + \dots + (m-1) \cdot |\alpha^{m-1}| = m. \end{cases}$$

Set

$$\mathbb{N}_0(\alpha, m) := \{(\alpha^1, \alpha^2, \dots, \alpha^{m-1}) \in \mathbb{N}(\alpha, m) : \alpha_1 = \alpha\},$$

$$\mathbb{N}_1(\alpha, m) := \{(\alpha^1, \alpha^2, \dots, \alpha^{m-1}) \in \mathbb{N}(\alpha, m) : |\alpha_1 - \alpha| = 1\},$$

$$\mathbb{N}^*(\alpha, m) := \{(\alpha^1, \alpha^2, \dots, \alpha^{m-1}) \in \mathbb{N}(\alpha, m) : |\alpha_1 - \alpha| \geq 2\},$$

and denote the corresponding sums in (6.11) by  $Q_\alpha^0(I)$ ,  $Q_\alpha^1(I)$  and  $Q_\alpha^*(I)$  respectively.

1. *Estimate of  $Q_\alpha^0(I)$ .* The set  $\mathbb{N}_0(\alpha, m)$  contains only one element,  $|\alpha| = m \geq 3$ , and by (6.10) we get

$$\begin{aligned} Q_\alpha^0(I) &\leq P_s(b_\alpha) \leq \tilde{L}_0(e^\rho L_1)^s (e^{\mu-1} L_2)^{m-1} \Gamma(\rho s + (\mu-1)(m-2)) \\ &\leq 2^{1-m} C_1^s C_2^{m-1} \Gamma(\rho s + (\mu-1)(m-2)) \end{aligned}$$

for  $C_2 \geq 2e^{\mu-1} L_2$  and  $C_1 = e^\rho L_1$ .

2. *Estimate of  $Q_\alpha^1(I)$ .* Notice that the cardinality of  $\mathbb{N}_1$  is  $\#\mathbb{N}_1(\alpha, m) \leq n$ . Indeed, if  $(\alpha^1, \alpha^2, \dots, \alpha^{m-1}) \in \mathbb{N}_1(\alpha, m)$ , then we have  $|\alpha^1| = |\alpha| - 1 \geq 1$  and  $|\alpha^2| + \dots + |\alpha^{m-1}| = 1$ . Hence,  $\alpha^k = 0$  for any  $k \neq 1, m - |\alpha| + 1$ , and  $|\alpha^k| = 1$  for  $k = m - |\alpha| + 1$ , which implies  $\#\mathbb{N}_1(\alpha, m) \leq n$ . Moreover,  $\alpha!/\alpha^1! \leq |\alpha|$ .

Fix  $C_1 = e^\rho L_1$  and  $C_2 \geq 2e^{\mu-1} L_2$ . Using Lemma 6.5 with  $p = 1$ ,  $m_1 = m - |\alpha| + 1$  and  $M_1 = m - |\alpha|$ , we get

$$\begin{aligned} Q_\alpha^1(I) &\leq |\alpha| K'_0 \tilde{L}_0 (e^{\mu-1} L_2)^{|\alpha|-1} C_1^{s+1} C_2^{m-|\alpha|} \Gamma(\rho s + (\mu-1)(m-2)) \\ &\leq |\alpha| (K'_0 \tilde{L}_0 L_1 L_2) (e^{\mu-1} L_2)^{|\alpha|-2} C_1^s C_2^{m-|\alpha|} \Gamma(\rho s + (\mu-1)(m-2)) \\ &\leq |\alpha| 2^{-|\alpha|} (K'_0 \tilde{L}_0 L_1 L_2) C_1^s C_2^{m-2} \Gamma(\rho s + (\mu-1)(m-2)), \end{aligned}$$

where  $K'_0 = K'_0(n, \rho, \mu)$  stands for different constants depending only on  $n, \rho$  and  $\mu$ .

3. *Estimate of  $Q_\alpha^*(I)$ .* Let  $(\alpha^1, \alpha^2, \dots, \alpha^{m-1}) \in \mathbb{N}^*(\alpha, m)$ . Set as above

$$F := \left( \frac{\partial g_2}{\partial \theta}(\cdot, I) \right)^{\alpha^2} \cdots \left( \frac{\partial g_{m-1}}{\partial \theta}(\cdot, I) \right)^{\alpha^{m-1}}.$$

Notice that the corresponding  $p$  in Lemma 6.5 is

$$p = |\alpha_2| + \cdots + |\alpha_{m-1}| \geq 2.$$

Moreover,  $p \leq |\alpha|$  and

$$\begin{aligned} M_p &:= 1 \cdot |\alpha_1| + 2 \cdot |\alpha_2| + \cdots + (m-1) \cdot |\alpha_{m-1}| \\ &- |\alpha_1 + \alpha_2 + \cdots + \alpha_{m-1}| = m - |\alpha| \geq 2. \end{aligned}$$

It follows from Lemma 6.5 and Remark 6.3 that

$$\begin{aligned} P_s(b_\alpha F) &\leq K_0 \tilde{L}_0 (C_0 e^{\mu-1} L_2)^{|\alpha|-1} C_1^{s+|\alpha|} C_2^{m-|\alpha|} \\ &\times \Gamma(\rho s + (\mu-1)(m-2)) \binom{m-2}{|\alpha|-2}^{-1} (m-|\alpha|)^{-1} \end{aligned}$$

for any  $I \in \mathbb{D}^n$ . We have

$$\binom{m-1}{|\alpha|-1} \binom{m-2}{|\alpha|-2}^{-1} (m-|\alpha|)^{-1} = \frac{m-1}{(|\alpha|-1)(m-|\alpha|)} \leq 2.$$

Then using Lemma 7.3 we estimate  $Q_\alpha^*(I)$  by

$$\begin{aligned} Q_\alpha^*(I) &\leq K_0 \tilde{L}_0 (C_0 e^{\mu-1} L_2)^{|\alpha|-1} C_1^{s+|\alpha|} C_2^{m-|\alpha|} \Gamma(\rho s + (\mu-1)(m-2)) \\ &\times \binom{m-1}{|\alpha|-1} \binom{m-2}{|\alpha|-2}^{-1} (m-|\alpha|)^{-1} \\ &\leq (K'_0 \tilde{L}_0 L_1^2 L_2) (C_0 e^\rho e^{\mu-1} L_1 L_2)^{|\alpha|-2} C_1^s C_2^{m-|\alpha|} \Gamma(\rho s + (\mu-1)(m-2)). \end{aligned}$$

Hereafter  $K'_0 = K'_0(n, \rho, \mu)$  stands for different constants depending only on  $n, \rho$  and  $\mu$ . For  $C_2 \geq 2C_0(\delta, \mu)e^\rho e^{\mu-1} L_1 L_2$  we obtain

$$Q_\alpha^*(I) \leq 2^{-|\alpha|} (K'_0 C_0 \tilde{L}_0 L_1^2 L_2) C_1^s C_2^{m-2} \Gamma(\rho s + (\mu-1)(m-2)).$$

Taking into account the cases 1. - 3. we obtain

$$Q_\alpha(I) \leq |\alpha| 2^{-|\alpha|} (K'_0 \tilde{L}_0 L_1^2 L_2) C_1^s C_2^{m-2} \Gamma(\rho s + (m-2)(\mu-1))$$

for any  $I \in \mathbb{D}^n$ , where

$$C_1 = e^\rho L_1 \text{ and } C_2 \geq 2C_0(\delta, \mu)e^\rho e^{\mu-1} L_1^2 L_2. \quad (6.12)$$

Set

$$B_0 := K'_0 \sum_{p=0}^{\infty} (p+1)^{n+1} 2^{-p}.$$

Then for any  $s \geq 0$ ,  $m \geq 3$ , and  $I \in \mathbb{D}^n$  we obtain

$$P_s(B_m(\cdot, I)) \leq B_0 \tilde{L}_0 L_1^2 L_2 C_1^s C_2^{m-2} \Gamma(\rho s + (\mu - 1)(m - 2)),$$

which proves (6.3).

This implies

$$|R_m(I)| \leq B_0 \tilde{L}_0 L_1^2 L_2 C_1^s C_2^{m-2} \Gamma(\rho s + (\mu - 1)(m - 2)).$$

Now (4.2) yields

$$\begin{aligned} P_s(g_m(\cdot, I)) &\leq \frac{1}{\kappa} P_{s+\tau}(B_m(\cdot, I)) \\ &\leq B_0 e^{\tau\rho} \frac{1}{\kappa} \tilde{L}_0 L_1^{\tau+2} L_2 C_1^s C_2^{m-2} \Gamma(\rho s + \rho\tau + (\mu - 1)(m - 2)) \end{aligned}$$

for any  $I \in \mathbb{D}^n$ . Set  $A_0 := \max\{B_0 e^{\tau\rho}, 2C_0(\delta, \mu)e^\rho e^{\mu-1}\}$  and fix

$$C_2 \geq \frac{1}{\kappa} A_0 \tilde{L}_0 L_1^{\tau+2} L_2.$$

Since  $\kappa \leq 1$  and  $\tilde{L}_0 \geq 1$  the inequality in (6.12) holds as well. As  $\mu = \rho(\tau + 1) + 1$ , we obtain

$$P_s(g_m(\cdot, I)) \leq C_1^s C_2^{m-1} \Gamma(\rho s + (\mu - 1)(m - 1) - \rho)$$

for any  $I \in \mathbb{D}^n$ . This completes the induction and proves Proposition 6.1.  $\square$

*Proof of Proposition 3.2.* By the Cauchy formula and Lemma 4.3 we get for any  $\alpha \in \mathbb{N}^n$  with  $|\alpha| \geq 2$  the estimate

$$\begin{aligned} \left| \partial_\theta^\beta g_{|\alpha|, \alpha}(\theta) \right| &= \frac{1}{\alpha!} \left| \partial_I^\alpha \partial_\theta^\beta g_{|\alpha|}(\theta, 0) \right| \\ &\leq \sup_{I \in \mathbb{D}^n} \left| \partial_\theta^\beta g_{|\alpha|}(\theta, I) \right| \leq \sup_{I \in \mathbb{D}^n} P_{|\beta|}(g_{|\alpha|}(\cdot, I)). \end{aligned}$$

Now Proposition 6.1 and (7.5) imply

$$\begin{aligned} \sup_{\theta \in \mathbb{T}^n} \left| \partial_\theta^\beta g_{|\alpha|, \alpha}(\theta) \right| &\leq C_1^{|\beta|} C_2^{|\alpha|-1} \Gamma(\rho|\beta| + (\mu - 1)(|\alpha| - 1) - \rho) \\ &\leq (cC_2)^{-1} (cC_1)^{|\beta|} (cC_2)^{|\alpha|} \Gamma(\rho|\beta| + 1) \Gamma((\mu - 1)|\alpha| + 1), \end{aligned}$$

for any  $\alpha, \beta \in \mathbb{N}^n$ , where  $c = c(\rho, \mu) \geq 1$ . Using the Borel extension theorem in Gevrey classes (see [18, Theorem 3.7] for a more general version) we find a  $\mathcal{G}^{\rho, \mu}$ -smooth function  $g$  such that  $g(\theta, I) \sim \sum_{m=2}^\infty \sum_{|\alpha|=m} g_{m, \alpha}(\theta) I^\alpha$ , i.e. the Taylor series of  $g$  is given by (5.1). Moreover, we have

$$\sup_{(\theta, I) \in \mathbb{A}'} |\partial_\theta^\beta \partial_I^\alpha g(\theta, I)| \leq \frac{C_0}{C_2} C_1^{|\beta|} C_2^{|\alpha|} \Gamma(\rho|\beta| + 1) \Gamma(\mu|\alpha| + 1), \quad (6.13)$$

for any  $\alpha, \beta \in \mathbb{N}^n$ , where  $\mathbb{A}' = \mathbb{T}^n \times D'$ ,  $D'$  is a neighborhood of 0 in  $\mathbb{R}^n$ ,  $C_0 = C_0(\rho, \tau, n) \geq 1$ , and the constants  $C_1 \geq 1$  and  $C_2 \geq 1$  are equivalent to  $L_1$  and  $\frac{1}{\kappa} \tilde{L}_0 L_1^{\tau+2} L_2$  respectively. In particular,  $g$  belongs

to  $\mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}')$  and  $\|g\|_{C_1, C_2} \leq C_0/C_2$ . In the same way we find  $H^0 \in \mathcal{G}_{C_2}^{\mu}(D')$  such that  $H^0(I) \sim \sum R_m(I)$ . Then, using the composition of Gevrey functions [18, Proposition A.4], we show that the function  $H'$  defined by  $H'(\theta, I) := H(\theta, I + \partial g / \partial \theta(\theta, I))$  belongs to  $\mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}')$ , where the Gevrey constants  $C_1$  and  $C_2$  are equivalent to  $L_1$  and to  $\frac{1}{\kappa} \tilde{L}_0 L_1^{\tau+2} L_2$  respectively. Recall that  $\tilde{L}_0$  is equivalent to  $L_0 L_1^{n+2}$ , hence,  $C_1$  and  $C_2$  satisfy (3.5). This completes the proof of Proposition 3.2.  $\square$

*Proof of Theorem 2.* We are going to solve the equation

$$\varphi = \theta + \nabla_I g(\theta, I), \quad (\theta, I) \in \mathbb{A}', \quad (6.14)$$

with respect to  $\theta \in \mathbb{T}^n$ , by means of the implicit function theorem in anisotropic Gevrey classes [18, Proposition A.2]. By (6.13) we have

$$\sup_{(\theta, I) \in \mathbb{T}^n \times D'} \|\partial_\theta^\beta \partial_I^\alpha \nabla_I g(\theta, I)\| \leq C_0 C_1^{|\beta|} C_2^{|\alpha|} \Gamma(\rho|\beta| + 1) \Gamma(\mu|\alpha| + 1) \quad (6.15)$$

where  $C_0$  is equivalent to 1,  $C_1$  is equivalent to  $L_1$ , and  $C_2$  is equivalent to  $\frac{1}{\kappa} L_0 L_1^{\tau+n+4} L_2$ . Set

$$\epsilon := (2C_0 C_1)^{-1} < 1 \quad \text{and} \quad \tilde{C}_2 := a(\rho, \mu, n) C_2 / \epsilon.$$

Then choosing  $D'$  small enough and  $a(\rho, \mu, n) \gg 1$  we obtain by (6.15)

$$\sup_{(\theta, I) \in \mathbb{T}^n \times D'} \|\partial_\theta^\beta \partial_I^\alpha \nabla_I g(\theta, I)\| \leq \epsilon C_0 C_1^{|\beta|} \tilde{C}_2^{|\alpha|} \Gamma(\rho|\beta| + 1) \Gamma(\mu|\alpha| + 1)$$

for any  $\alpha, \beta \in \mathbb{N}^n$ . Now,  $\epsilon C_0 C_1 < 1/2$  and [18, Proposition A.2] implies that there is  $\theta \in \mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}', \mathbb{T}^n)$  which solves (6.14), where

$$C_1 = c(\rho, \tau, n) L_1 \quad \text{and} \quad C_2 = c(\rho, \tau, n) \frac{1}{\kappa} L_1^{\tau+n+5} L_2.$$

Next using the theorem of composition of Gevrey functions [18, Proposition A.4], we prove that  $R^0(\varphi, I) := R(\theta(\varphi, I), I)$  belongs to the class  $\mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}', \mathbb{A})$ , where  $C_1$  and  $C_2$  are given by Remark 3.1. By the same argument, the canonical transformation  $\chi$  generated by  $g$  belongs to the class  $\mathcal{G}_{C_1, C_2}^{\rho, \mu}(\mathbb{A}', \mathbb{A})$ . This completes the proof of Theorem 2 and of Remark 3.1.

## 7. COMPLEMENTS ON THE GAMMA FUNCTION

Here we collect certain estimates of the Euler Gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0, \quad (7.1)$$

that have been used above. Recall that  $\Gamma(x+1) = x\Gamma(x)$  which implies  $\Gamma(m+1) = m!$  for any  $m \in \mathbb{N}$ . Moreover,  $\Gamma(t)$  is convex in the interval  $(0, +\infty)$ , it has a minimum at some point  $t_0 \approx 1,46$  and  $\Gamma(t_0) \approx 0,89$ .

In particular,  $\Gamma(t)$  is strictly decreasing in  $(0, t_0]$  and strictly increasing in  $[t_0, +\infty)$ . We have the following relation (see [2], [15])

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x, y), \quad x, y > 0, \quad (7.2)$$

where  $B(x, y)$  is the Beta function which is defined for  $x > 0$  and  $y > 0$  by the following integral representation

$$B(x, y) = \int_0^1 (1-t)^{x-1} t^{y-1} dt. \quad (7.3)$$

Obviously,  $B(x, y)$  is symmetric, i.e.  $B(x, y) = B(y, x)$ , and it is decreasing with respect to both variables  $x$  and  $y$ . Using the integral representation in (7.3) and the Cauchy inequality we get for any positive numbers  $a, b, c, d$  the following inequality

$$B(a+b, c+d) \leq B(2a, 2c)^{1/2} B(2b, 2d)^{1/2}. \quad (7.4)$$

Denote by  $[x]$  the entire part of  $x \in \mathbb{R}$ . Since  $B(x, y)$  is a decreasing function with respect to both variables  $x, y > 0$ , we have

$$B(x, y) \geq B([x] + 1, [y] + 1) = \frac{[x]![y]!}{([x] + [y] + 1)!} \geq 4^{-x-y}. \quad (7.5)$$

For any  $x, y \geq 0$  we get in the same way

$$\begin{aligned} B(x+1, y+1) &\leq B([x] + 1, [y] + 1) = \frac{1}{[x] + [y] + 1} \binom{[x] + [y]}{[y]}^{-1} \\ &< \frac{3}{x + y + 1} \binom{[x] + [y]}{[y]}^{-1}. \end{aligned}$$

More generally, we have the following

**Lemma 7.1.** *For any  $\nu \geq 1$  and  $\delta > 0$  there is a constant  $C'(\nu, \delta) \geq 1$  such that for any  $x, y \geq 0$  the following inequality holds*

$$\left( \frac{[x] + [y]}{[x]} \right)^\nu B(\nu x + \delta, \nu y + \delta) \leq \frac{C'(\nu, \delta)}{(\min(x+1, y+1))^{(\nu+1)/2}}.$$

*Proof.* Since  $B(x, y)$  is a decreasing function with respect to both variables  $x > 0$  and  $y > 0$  we can suppose that  $\delta \leq 1$ . Fix  $0 < \epsilon \leq 1$ . By Stirling's formula and the continuity of the Gamma function in the interval  $[\epsilon, +\infty)$ , there is  $L = L(\epsilon) > 1$  such that for any  $t \geq \epsilon$  we have

$$L^{-1} \leq \Gamma(t)(2\pi)^{-1/2} t^{\frac{1}{2}-t} e^t \leq L.$$

For any  $\nu \geq 1$  and  $t \geq \epsilon$  this implies the two-sided inequality

$$L^{-\nu-1} \left( \frac{t}{2\pi} \right)^{(\nu-1)/2} \nu^{\nu t - \frac{1}{2}} \leq \frac{\Gamma(\nu t)}{\Gamma(t)^\nu} \leq L^{\nu+1} \left( \frac{t}{2\pi} \right)^{(\nu-1)/2} \nu^{\nu t - \frac{1}{2}}. \quad (7.6)$$



Set  $\epsilon := \delta/\nu \in (0, 1]$ . Substituting  $t = x + \epsilon$ ,  $t = y + \epsilon$  and  $t = x + y + 2\epsilon$  in (7.6), where  $x \geq 0$  and  $y \geq 0$ , we obtain

$$\begin{aligned} B(\nu x + \delta, \nu y + \delta) &= \frac{\Gamma(\nu(x + \epsilon))\Gamma(\nu(y + \epsilon))}{\Gamma(\nu(x + y + 2\epsilon))} \\ &\leq C \left( \frac{(x + \epsilon)(y + \epsilon)}{x + y + 2\epsilon} \right)^{\frac{\nu-1}{2}} B(x + \epsilon, y + \epsilon)^\nu \\ &= C (x + y + 1 + 2\epsilon)^{(\nu-1)/2} B(x + 1 + \epsilon, y + 1 + \epsilon)^{(\nu-1)/2} \\ &\quad \times B(x + \epsilon, y + \epsilon)^{(\nu+1)/2}, \end{aligned}$$

where  $C = L^{3\nu+3}(2\pi)^{(1-\nu)/2}\nu^{-1/2}$ . For  $y \geq x \geq 0$  we have

$$\begin{aligned} B(x + \epsilon, y + \epsilon) &\leq \frac{(x + y + 2\epsilon)(x + y + 1 + 2\epsilon)}{(x + \epsilon)(y + \epsilon)} B(x + 1 + \epsilon, y + 1 + \epsilon) \\ &\leq \frac{2(x + y + 1 + 2\epsilon)}{x + \epsilon} B(x + 1 + \epsilon, y + 1 + \epsilon), \end{aligned}$$

which implies

$$\begin{aligned} B(\nu x + \delta, \nu y + \delta) &\leq C 2^{(\nu+1)/2} \frac{(x + y + 1 + 2\epsilon)^\nu}{(x + \epsilon)^{(\nu+1)/2}} \\ &\quad \times B(x + 1 + \epsilon, y + 1 + \epsilon)^\nu. \end{aligned}$$

Since  $B(x, y)$  is a decreasing function with respect to both variables  $x$  and  $y$ , we obtain

$$\begin{aligned} B(x + 1 + \epsilon, y + 1 + \epsilon) &\leq B([x] + 1, [y] + 1) \\ &= \frac{1}{[x] + [y] + 1} \binom{[x] + [y]}{[y]}^{-1} \leq \frac{5}{x + y + 1 + 2\epsilon} \binom{[x] + [y]}{[y]}^{-1}. \end{aligned}$$

This implies

$$B(\nu x + \delta, \nu y + \delta) \leq \frac{C'}{(x + 1)^{(\nu+1)/2}} \binom{[x] + [y]}{[x]}^{-\nu}.$$

where  $C' = \left(\frac{2\nu}{\delta}\right)^{(\nu+1)/2} 5^\nu C$ . This completes the proof of the assertion since the inequality in Lemma 7.1 is symmetric with respect to  $x, y$ .  $\square$

**Remark 7.2.** As in (7.6) one proves that for any  $\rho > 0$  there is a constant  $C(\rho) > 1$  such that

$$C(\rho)^{-m} \Gamma(\rho m + 1) \leq m!^\rho \leq C(\rho)^m \Gamma(\rho m + 1)$$

for any  $m \in \mathbb{N}$ .

We have also

**Lemma 7.3.** *For any  $m \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^n$  such that  $2 \leq |\alpha| \leq m$  we have*

$$\sum_{(\alpha^1, \dots, \alpha^m) \in \tilde{\mathbb{N}}(\alpha, m)} \frac{\alpha!}{\alpha^1! \dots \alpha^m!} = \frac{(m-1)!}{(m-|\alpha|)! (|\alpha|-1)!},$$

where  $\mathbb{N}(\alpha, m)$  is defined by (5.2).

*Proof.* Let  $f$  be an analytic function in  $\mathbb{R}^n$ . Then the left hand-side of the inequality above coincides with the coefficient  $a_{\alpha, m}$  in the identity

$$\begin{aligned} f\left(\frac{X}{1-X}, \dots, \frac{X}{1-X}\right) &= \sum_{\gamma \in \mathbb{N}^n} \left( \sum_{k=1}^{\infty} X^k, \dots, \sum_{k=1}^{\infty} X^k \right)^{\gamma} \frac{f^{(\gamma)}(0)}{\gamma!} \\ &= \sum_{\gamma \in \mathbb{N}^n} \sum_{\nu \geq |\gamma|} a_{\gamma, \nu} X^{\nu} \frac{f^{(\gamma)}(0)}{\gamma!}. \end{aligned}$$

Now taking  $f(Y) = Y^{\alpha}$  we obtain

$$a_{\alpha, m} = \frac{1}{m!} \left( \frac{d}{dX} \right)^m (X^{|\alpha|} (1-X)^{-|\alpha|})|_{X=0} = \frac{(m-1)!}{(m-|\alpha|)! (|\alpha|-1)!}. \quad \square$$

**Acknowledgments.** *Part of this work has been done at the Institute of Mathematics, Bulgarian Academy of Sciences, and I would like to thank the colleagues there for the stimulating discussions.*

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